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The non-commutative brane world

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Abstract: We propose a new higher-dimensional mechanism to localize scalar fields as well as fermionic and gauge fields. The underlying theory is a six-dimensional non-commutative field theory where non-commutativity is allowed along two extra infinite spatial dimensions and the four-dimensional brane is provided by a scalar soliton living in the non-commutative space. Making use of the powerful correspondence between non-commutative coordinates and operators on a single particle Hilbert space, we show that the non-commutative brane world admits localized chiral fermions and it ensures the localization of massless gauge fields. It may also give rise to a variety of different low-energy spectra since the localized zero mode may come along either with a discrete tower of degenerate heavy states or with a tower of Kaluza-Klein heavy states, or it may even be the only state in the low-energy spectrum.

Keywords: Field Theories in Higher Dimensions, Non-Commutative Geometry.

Contents

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1. Introduction

The capability of confining Standard Model fields on $(3+1)$-dimensional subspaces (branes) [1] of a higher-dimensional manifold has recently given rise to scenarios where the size of the extra dimensions may be much larger than the tiny four-dimensional Planck length. In models where the space-time geometry is of a simple factorizable form, the space of extra dimensions — the bulk — may be compact and perhaps as large as a millimeter [2]. If the space-time geometry has a non-factorizable form, the extra dimensions may be warped and non-compact [3]. In particular, the possibility of warped non-compact extra dimensions has extended our intuition about how extra spatial dimensions are manifest in four-dimensional effective field theories by showing that even if gravity propagates in non-compact higher dimensional spaces, four-dimensional observers may still empirically deduce a four-dimensional Newton's law.

In this paper we will show that the four-dimensional localization of scalar, fermionic and gauge fields may occur in a six-dimensional non-commutative field theory where the non-commutativity is present in two extra non-compact spatial dimensions. Indeed, we find that the magic of non-commutativity facilitates by a great deal the study of localization of fields and may give rise to many and different kinds of spectra in the low-energy four-dimensional theory.

Non-commutative field theories have turned out to be very relevant for a broad variety of unexpected applications. They are nonlocal field theories where locality breaks down at short distances and they might provide new insights into the issue of non-locality in quantum gravity. The study of the perturbative aspects of non-commutative field theories has revealed an intriguing mixing of the UV and IR arising from non-planar divergent diagrams which are interpreted as IR divergences [4]. It has also been understood that non-commutative gauge theories arise from a limit of string theory [5]–[8].

More recently, soliton solutions of scalar non-commutative field theories have been constructed [9]. These solutions are solitons whose size is set by the scale of non-commutativity and play an important role in constructing D-branes as non-commutative solitons of the tachyon field of open string theory [10, 13, 14].

We start from a non-commutative field theory of a single real scalar Φ in a six dimensional space where non-commutativity is present along the two extra infinite spatial dimensions, i.e. $M_4 \times \mathbb{R}^2_\star$, where M_4 denotes the four-dimensional Minkowski space-time. The four-dimensional brane is given by a radially symmetric soliton in $\mathbb{R}^2_\star$ and we study the localization of fields on such a brane taking the limit in which the non-commutative length is much larger than the inverse of the fundamental mass scale in 6D. The 6D action is initially invariant under a global $\mathrm{U}(\infty)$ symmetry which is broken down to $\mathrm{U}(\infty-1)\otimes\mathrm{U}(1)$ by the soliton.

Our construction of the non-commutative brane world exploits the powerful connection between non-commutative coordinates and operators in single particle quantum mechanics. Under this correspondence, the $\star$-product (the generalization in non-commutative spaces of the usual product) maps onto usual operator multiplication and the equation of motions translates into algebraic operator equations. The four-dimensional action and the corresponding low-energy spectrum can be easily computed by carrying out a the trace over operators and computations are facilitated by the fact that the non-commutative soliton Φ_0 acts like a projector operator in the Hilbert space of the single particle quantum mechanics. Fields interacting with the soliton may be decomposed from the four-dimensional point of view along a a complete and discrete set of orthogonal functions in $\mathbb{R}^2_\star$ and the presence of the soliton projects out most of the unwanted modes, leaving behind a massless mode.

A remarkable feature of the resulting four-dimensional action describing the dynamics of the localized zero modes is its insensitivity to the details of the underlying non-commutative field theory giving rise to it. The localization of a generic field interacting with the non-commutative soliton is due entirely to its six-dimensional kinetic term and to the fact that the soliton acts like a four-dimensional warp-factor. The 4D action inherits informations about the non-commutative soliton only in the point where the soliton field is maximized. This sounds quite intriguing if we think that the soliton is much broader than the inverse of the fundamental mass scale of the theory.

Depending upon the kind of interaction between a given six-dimensional field and the scalar soliton, different four-dimensional spectra may arise. In particular, the localized zero mode of a scalar field may come with either a discrete tower of degenerate heavy states or with a tower of Kaluza-Klein heavy states, or it may be the only state in the low-energy spectrum! We will also show that the non-commutative brane world admits localized chiral fermions and ensures the localization of massless gauge fields once the initial global $U(\infty)$ symmetry is promoted to a non-commutative $U(1)$ gauge symmetry.

The paper is organized as follows. In section 2 we first review a few crucial properties of non-commutative field theories and their solitons. In section 3 we present our results for the localization of scalar fields, while localization of fermions is studied in section 4. In section 5 we see how gauge massless fields descend from the soliton world-volume. Finally, in section 6 we present our conclusions and comment about possible directions for future work.

2. Some generalities

2.1 The non-commutative soliton

As we announced in the introduction, our starting point is a non-commutative field theory of a single real scalar Φ in a six dimensional space with non-commutativity in the two extra infinite spatial dimensions, i.e. $M_4 \times \mathbb{R}^2_\star$.

We will denote the complete set of space-time coordinates by x^μ $(\mu = 0, \ldots, 5)$, the non-commutative spatial directions by x^i $(i = 4, 5)$ and the remaining directions by x^a $(a = 0, \ldots, 3)$.

The spatial coordinates x^4 and x^5 commute according to

$$\left[x^4, x^5\right] = i\,\Theta \tag{2.1}$$

and can be parametrized in terms of the complex coordinates $z = (x^4 + ix^5)/\sqrt{2}$ and $\bar{z} = (x^4 - ix^5)/\sqrt{2}$.

Fields in the non-local action are intended to be multiplied using the Moyal star product,

$$(f \star g)\,(z, \bar{z}) \;=\; e^{(\Theta/2)(\partial_z \partial_{\bar{z}'} - \partial_{z'}\partial_{\bar{z}})} f(z, \bar{z}) g(z', \bar{z}')\,\big|_{z=z'}\;. \tag{2.2}$$

There is a useful one-to-one correspondence between functions $f(z, \bar{z})$ on the non-commutative $\mathbb{R}^2_\star$, thought of as the phase space of a particle in one dimension, and operators $O_f(\hat{z}, \hat{\bar{z}})$ acting on the Hilbert space $\mathcal{H}$ of a quantum system with one degree of freedom. Multiplication by the $\star$-product goes over to operator multiplication, and integration over $\mathbb{R}^2_\star$ corresponds to taking the trace over the Hilbert space,

$$f \star g \leftrightarrow \hat{A}\hat{B}\,, \qquad \frac{1}{2\pi\Theta}\int d^2z\, f(z,\bar{z}) \leftrightarrow \mathrm{Tr}\; O_f(\hat{z},\hat{\bar{z}})\,. \tag{2.3}$$

With this prescription the Moyal product of functions is isomorphic to ordinary operator multiplication

$$O_f \cdot O_g \leftrightarrow O_{f\star g}\,. \tag{2.4}$$

Consider now the action of a real scalar field Φ in $\mathbb{R}^6_\star$

$$S = \int d^6x \left[\frac{1}{2}\,\partial_\mu\Phi\,\partial^\mu\Phi \;-\; V(\Phi)\right]\,. \tag{2.5}$$

The correspondence (2.3) was exploited in [9] to construct general soliton solutions in $\mathbb{R}^2_\star$ in the limit of large non-commutativity. If we render the coordinates z and $\bar{z}$ dimensionless through the scaling

$$z = w\sqrt{\Theta}\,, \qquad \bar{z} = \bar{w}\,\sqrt{\Theta}\,, \tag{2.6}$$

the $\star$-product (2.2) will have no Θ, i.e. it will be given by (2.2) with $\Theta = 1$. Written in rescaled coordinates, the dependence on Θ in the two-dimensional part of the action is entirely in front of the potential term

$$\int d^2w \left[-\frac{1}{2}\partial_w\Phi\partial_{\bar{w}}\Phi - \Theta\, V(\Phi)\right]\,. \tag{2.7}$$

In the limit $\Theta \to \infty$[1] with $V(\Phi)$ held fixed, the kinetic term is negligible compared to $V(\Phi)$, at least for field configurations varying over sizes of order one in the new coordinates.

The equation of motion for static solitons is therefore

$$\frac{dV(\Phi)}{d\Phi} = 0\,. \tag{2.8}$$

Localized soliton solutions to this equation exist due to the presence of the $\star$-product [9]. The construction relies on the existence of functions Φ satisfying

$$\Phi_0 \star \Phi_0 = \Phi_0\,. \tag{2.9}$$

In such a case, the following crucial property holds

$$f(\phi_*\Phi_0) = f(\phi_*)\Phi_0\,, \tag{2.10}$$

[1]This limit should be intended as $\Theta M_f^2 \to \infty$ where M_f is the fundamental scale of the 6D-theory.

for any function f of the form $f(\Phi_0) = \sum_{n=1}^{\infty} a_n \Phi_0^n$. In particular,

$$\frac{dV(\Phi)}{d\Phi}|_{\Phi=\phi_*\Phi_0} = \left(\frac{dV(\Phi)}{d\Phi}|_{\Phi=\phi_*}\right)\Phi_0 \tag{2.11}$$

and eq. (2.8) is solved by choosing ϕ_* to be an extremum of $V(\Phi)$.

The simplest function satisfying (2.9) is the gaussian (in the non-rescaled coordinates)

$$\Phi_0(r) = 2e^{-r^2/\Theta}\,, \qquad r^2 = x_4^2 + x_5^2 \tag{2.12}$$

and the soliton solution is

$$\Phi = \phi_*\ \Phi_0(r)\,, \tag{2.13}$$

where ϕ_* is an extremum of $V(\Phi)$, $V'(\phi_*) = 0$. The non-locality of the $\star$-product makes it possible for a lump of approximately size $\sqrt{\Theta}$ to square itself under the $\star$-product and the resulting object is a $3+1$ dimensional soliton that we will identify with the brane world where matter fields live. Note that the soliton solution asymptotically approaches the value $\Phi = 0$.

The soliton solution is stable [9]. Indeed, suppose there are two solutions to $V'(\phi_*) = 0$, one for $(\phi_*)_1 = 0$ and $V[(\phi_*)_1] = 0$ and one for $(\phi_*)_1$ and $V[(\phi_*)_2] \neq 0$ and suppose they are separated by a finite barrier. Far away from the origin, $\Phi_0(r) = 0$, but near $r = 0$, $\Phi_0(r)$ is in the vicinity of the second vacuum, $\phi_* = (\phi_*)_2$. The solution corresponds to a bubble of false vacuum. The area of the bubble is of order one (or Θ in the non-rescaled coordinates). In a commutative theory such a bubble would decay by shrinking to zero size. Non-commutativity prevents the bubble from shrinking to a spatial size smaller than $\sqrt{\Theta}$. In order to decay, Φ_0 would have to scale to zero, a process which is classically forbidden [9]. Of course, the soliton is unstable against quantum decay, but the decay rate can be made arbitrarily small by adjusting the parameters of the potential.

2.2 The symmetries of the problem

The soliton solution $\Phi = \phi_*\Phi_0$ with $V'(\phi_*) = 0$ corresponds to the projection operator P_0 onto the ground state of a one-dimensional harmonic oscillator

$$\Phi_0 \sim P_0 = |0\rangle\langle 0| \tag{2.14}$$

and is not the unique solution to the equation $\Phi\star\Phi = \Phi$ (or, in terms of the generic projector operator P, $P^2 = P$). Other solutions may be obtained by choosing other projection operators [9]

$$\Phi_n \sim P_n = |n\rangle\langle n|\,, \tag{2.15}$$

or we can choose a superposition of solutions (a level k solution in the terminology of [9])

$$\Phi_k = \phi_*(\Phi_0 + \Phi_1 + \ldots + \Phi_{k-1})\,. \tag{2.16}$$

Since the $\phi_m = |m\rangle\langle m|$ are a complete set of projection operators, in the limit $k \to \infty$ $\Phi_\infty = \phi_* \mathbf{1}$. The corresponding soliton takes the value $\Phi = \phi_*$ everywhere, and no spatial structure is present in $\mathbb{R}^2_\star$.

In the limit of infinite non-commutativity the action (2.5) can be written in the operator form

$$S = \int d^4x \ \mathrm{Tr} \left[\frac{1}{2}\partial_a \hat{\Phi} \partial^a \hat{\Phi} - V(\hat{\Phi})\right], \tag{2.17}$$

where x^a are the commutative directions. The action is manifestly invariant under a global $\mathrm{U}(\infty)$ symmetry

$$\hat{\Phi} \to U \hat{\Phi} U^\dagger \,. \tag{2.18}$$

In other words, if $\hat{\Phi}$ is a solution to the equations of motion — in the limit of large non-commutativity — so is $U\hat{\Phi}U^\dagger$, where U is any unitary operator acting on $\mathcal{H}$. A general hermitian operator $\hat{\Phi}$ (corresponding to a real field Φ) may be obtained by acting on a diagonal operator (which corresponds to a radially symmetric field configuration) by an element of the $\mathrm{U}(\infty)$ symmetry group. Thus every solution to $V'(\Phi) = 0$ may be obtained from a radially symmetric solution by means of $\mathrm{U}(\infty)$ symmetry transformations and the solutions consist of disjoint infinite dimensional manifolds labeled by the set of eigenvalues of the corresponding operator.

The $\mathrm{U}(\infty)$ symmetry is broken by any non-vanishing solution of $\Phi(r)$ of eq. (2.8). Consequently, every non-vanishing solution implies a number of exact massless Goldstone modes corresponding to small displacements about $\Phi(r)$ on the manifold of solutions. The generators of $\mathrm{U}(\infty)$ are given by

$$R_{nm} = |n\rangle\langle m| + |m\rangle\langle n| \tag{2.19}$$

and

$$S_{nm} = i(|n\rangle\langle m| - |m\rangle\langle n|) \,. \tag{2.20}$$

The Goldstone modes are therefore given by the nonzero elements of

$$\delta\Phi \propto [R_{nm}, \Phi] \,, \qquad [S_{nm}, \, \Phi] \,. \tag{2.21}$$

Since in the operator language the complex coordinates $z = (x^4 + ix^5)/\sqrt{2}$ and $\bar{z} = (x^4 - ix^5)/\sqrt{2}$ correspond respectively to the annihilation operator $a = z/\sqrt{\Theta}$ and destruction operator $\bar{a} = \bar{z}/\sqrt{\Theta}$, and

$$[a, \bar{a}] = 1 \,, \tag{2.22}$$

the generator of the translations along the coordinate z ($\bar{z}$) is the operator $\bar{a}$ (a), so that derivatives become

$$\partial_z = -\Theta^{-1/2}[\bar{a}, \cdot], \quad \partial_{\bar{z}} = \Theta^{-1/2}[a, \cdot]. \tag{2.23}$$

Since

$$[R_{nm}, |0\rangle\langle 0|] = \delta_{n0}\left[|m\rangle\langle 0| - |0\rangle\langle m|\right] - \delta_{m0}\left[|0\rangle\langle n| - |n\rangle\langle 0|\right], \tag{2.24}$$

and

$$[S_{nm}, |0\rangle\langle 0|] = i\delta_{n0}\left[|m\rangle\langle 0| + |0\rangle\langle m|\right] - i\delta_{m0}\left[|0\rangle\langle n| + |n\rangle\langle 0|\right], \tag{2.25}$$

the unbroken generators in the Φ_0-background are given by R_{00} and all the R_{nm} and S_{nm} with $m, n \geq 1$. Notice in particular that the linear combination $\sum_{n\geq 0}(2n{+}1)R_{nn}$ reproduces the generator $(a\bar{a}{+}\bar{a}a)$ of rotations in $\mathbb{R}^2_\star$. On the contrary, the generators R_{0n} and S_{n0} with $n \geq 1$ are broken in the Φ_0 background and give rise to an infinite tower of Goldstone modes $\{\varphi_{0n}, \varphi_{n0}\}$. The soliton solution Φ_0 therefore breaks the global symmetry $\mathrm{U}(\infty)$ down to $\mathrm{U}(\infty - 1) \otimes \mathrm{U}(1)$.

All these considerations refer to the case of infinite non-commutativity. At finite Θ, the kinetic term in action (2.5) explicitly breaks the $\mathrm{U}(\infty)$ symmetry down to ISO(2), the euclidean group in the two extra dimensions. Finite Θ effects will therefore lift the $\Theta \to \infty$ manifold of solutions to a discrete set of solutions, still guaranteeing the existence of at least one classically stable soliton, which is identifiable with the the gaussian $\phi_*\Phi_0(r^2)$ [9]. Finite Θ effects are also expected to provide masses to the (unwanted) Goldstone modes $\{\varphi_{0n}, \varphi_{n0}\}$. Alternatively, one can promote the $\mathrm{U}(\infty)$ symmetry to a gauge symmetry with the components of the gauge fields transverse to the soliton behaving as adjoint scalar fields on the soliton [13]. As we shall see, this gauge symmetry removes from the spectrum the Goldstone bosons of the soliton through the Higgs mechanism, leaving behind a massless gauge field in the 4D action.

Our strategy is therefore the following. We will work in the limit 1 TeV $\ll 1/\sqrt{\Theta} \ll M_f$, where M_f is the fundamental scale of the 6D theory. This range of Θ allows the presence of the stable soliton Φ_0 and at the same time to treat perturbatively the kinetic terms along the non-commutative directions in the action of matter fields and gauge fields coupled to the soliton.

3. Fluctuations around the soliton

In the rescaled coordinates the action of the real scalar field Φ is given by

$$S[\Phi] = \int d^6x\, \Theta \left[\frac{1}{2}\partial_a\Phi\partial^a\Phi + \Theta^{-1}\frac{1}{2}\partial_i\Phi \star \partial^i\Phi - V(\Phi)\right]. \tag{3.1}$$

Let ϕ_* be a real root of $V'(\phi_*) = 0$. In the limit of large Θ a non trivial static solution is $\Phi = \phi_*\Phi_0 \sim |0\rangle\langle 0|$, with $\Phi_0 \star \Phi_0 = \Phi_0$.

To find the 4D action, we apply some of the powerful simplifications following from non-commutative geometry. Using the operator correspondence, a complete set of functions in $\mathbb{R}^2_\star$ is given by

$$\Phi_{mn}(x^i) \sim |m\rangle\langle n|, \tag{3.2}$$

so the general fluctuation around the soliton Φ_0 is

$$\Phi = \phi_* \Phi_0(x^i) + \varphi\,, \qquad \varphi = \sum_{m,n=0} \varphi_{mn}(x^a)\Phi_{mn}(x^i)\,. \tag{3.3}$$

Consider $V(\Phi)$ of the form $V(\Phi) = c_2\Phi^2 - c_3\Phi^3 + c_4\Phi^4$, where all c_i's are positive and we do not allow a linear term (otherwise $\Phi = \phi_*\Phi_0$ cannot be a solution of eq.(2.8)). The extremum condition reads $2c_2\phi_* - 3c_3\phi_*^2 + 4c_4\phi_*^3 = 0$. Suppose the c_i's are such that $V'(\phi_*) = 0$, $V''(\phi_*) > 0$ for some non-vanishing ϕ_*. Performing the traces, one gets

$$\begin{aligned}
\mathrm{Tr}\left[(\phi_*\Phi_0 + \varphi)^2\right] &= \phi_*^2 + 2\phi_*\varphi_{00} + \sum_{m,n} \varphi_{mn}\,\varphi_{nm}\,, \\
\mathrm{Tr}\left[(\phi_*\Phi_0 + \varphi)^3\right] &= 3\phi_* \sum_m \varphi_{m0}\varphi_{0m} + 3\phi_*^2\varphi_{00} + \phi_*^3 + \mathcal{O}(\varphi^3)\,, \\
\mathrm{Tr}\left[(\phi_*\Phi_0 + \varphi)^4\right] &= 4\phi_*^2 \sum_m \varphi_{m0}\varphi_{0m} + 2\phi_*^2\varphi_{00} + 4\phi_*^3\varphi_{00} + \phi_*^4 + \mathcal{O}(\varphi^3)\,.
\end{aligned} \tag{3.4}$$

Making use of the operator correspondence (2.3) and the fact that $\Phi_0 \sim |0\rangle\langle 0|$ is orthogonal to Φ_{mn} for $m, n > 0$, one gets the following 4D effective action

$$\begin{aligned}
S[\phi_*\Phi_0 + \varphi] = \pi\,\Theta \int d^4x \Big\{ &\frac{1}{2}\partial_a\varphi_{00}\partial^a\varphi_{00} - \frac{1}{2}V''(\phi_*)\varphi_{00}^2 + V(\phi_*) + \\
&+ \sum_{m>0,n>0} \left(\frac{1}{2}\partial_a\varphi_{mn}\,\partial^a\varphi_{nm} - c_2\varphi_{mn}\,\varphi_{nm}\right) + \\
&+ \sum_{m>0} \left[\partial_a\varphi_{m0}\partial^a\varphi_{0m} - \left(2c_2 - 3\phi_* c_3 + 4c_4\phi_*^2\right)\varphi_{m0}\varphi_{0m}\right] \Big\}.
\end{aligned} \tag{3.5}$$

Taking into account that ϕ_* is an extremum, we are left with

$$\begin{aligned}
S[\phi_*\Phi_0 + \varphi] = \pi\Theta \int d^4x \Big[&\frac{1}{2}\partial_a\varphi_{00}\partial^a\varphi_{00} - \frac{1}{2}V''(\phi_*)\varphi_{00}^2 + V(\phi_*) + \\
&+ \sum_{m>0,n>0} \left(\frac{1}{2}\partial_a\varphi_{mn}\partial^a\varphi_{nm} - c_2\varphi_{mn}\varphi_{nm}\right) + \sum_{m>0} \partial_a\varphi_{m0}\partial^a\varphi_{0m} \Big]\,.
\end{aligned} \tag{3.6}$$

The four-dimensional action is therefore given by the heavy mode φ_{00} with mass squared $V''(\phi_*)$ (of the order of M_f^2), by the tower of heavy states φ_{nm} with mass $2c_2$ (again of the order of M_f^2), and by the announced tower of Goldstone modes $\{\varphi_{0n}, \varphi_{n0}\}$ (notice that $\varphi_{0n} = \varphi_{n0}^*$ because of the reality of the field Φ). We will see in the following how these unwanted Goldstone modes disappear from the 4D effective action once the U(∞) symmetry is gauged or — alternatively — when finite Θ-effects are taken into account.[2]

[2]The analysis can be easily repeated with a more general solution of eq. (2.8). In general, the soliton solution will contain operators of the form $|m\rangle\langle n|$, $m \neq n$, corresponding to non-spherically symmetric soliton fluctuations in the four-dimensional action.

4. Localization of scalar fields

Let us now consider a real scalar field χ coupled to the non-commutative soliton Φ_0 in the 6D theory. We will show that — according to the type of interaction we allow for the field χ with the soliton — different four-dimensional spectra may arise. In particular, we may obtain a four-dimensional action containing: either *i)* a zero mode together with a tower of *degenerate* heavy states, or *ii)* a zero mode with a tower of Kaluza-Klein heavy states, or *iii)* only a zero mode! Let us see how this comes about.

4.1 The four-dimensional action with a zero mode and a tower of degenerate heavy states

Let us consider the following action for the scalar field χ

$$\begin{aligned} S &= \int d^6x \left[\frac{1}{2} f(\Phi) \star \partial_\mu \chi \star \partial^\mu \chi - \frac{1}{2} m^2(\Phi) \star \chi \star g(\phi_* - \Phi) \star \chi\right] \qquad (4.1) \\ &= \int d^4x d^2z \left\{\frac{1}{2} f(\Phi) \star [\partial_a \chi \star \partial^a \chi - \partial_z \chi \star \partial_{\bar{z}} \chi - \partial_{\bar{z}} \chi \star \partial_z \chi] - \right. \\ &\qquad \left. - \frac{1}{2} m^2(\Phi) \star \chi \star g(\phi_* - \Phi) \star \chi \right\} ; \end{aligned}$$

with $f(0) = m^2(0) = g(0) = 0$. We stress that in (4.1) — due to the lack of commutativity — the derivatives $\partial_z \chi \star \partial_{\bar{z}} \chi$ and $\partial_{\bar{z}} \chi \star \partial_z \chi$ do not sum up. As for the fluctuations around the soliton, we can expand the field χ as

$$\chi(x^\mu) = \sum_{m,n \geq 0} \chi_{mn}(x^a) \Phi_{mn}(x^i) . \qquad (4.2)$$

The 4D effective action is obtained integrating out the non-commutative coordinates and setting $\Phi = \phi_* \Phi_0$. In the limit of infinite non-commutative parameter, the kinetic term from the extra dimensions is negligible. Since Φ_0 is a projector operator, when $\Phi = \phi_* \Phi_0$, we may write the crucial relation

$$g(\phi_* - \Phi) = g(\phi_*) \, (1 - \Phi_0) . \qquad (4.3)$$

This will have the effect of projecting out the zero mode χ_{00} from the mass term. Indeed — integrating over the two extra-dimensions — the 4D action reads

$$\begin{aligned} S = \pi \Theta \int d^4x \Bigg\{ & \frac{1}{2} f(\phi_*) \partial_a \chi_{00} \partial^a \chi_{00} + \sum_{n \geq 1} \Bigg[\frac{1}{2} f(\phi_*) \partial_a \chi_{0n} \partial_a \chi_{n0} - \\ & - \frac{1}{2} m^2(\phi_*) g(\phi_*) \chi_{n0} \chi_{0n} \Bigg] \Bigg\} . \end{aligned} \qquad (4.4)$$

The fluctuations $\chi_{mn}(x^a)$ with $m, m \geq 1$ have been projected out since they are orthogonal to the soliton $\Phi_0 \sim |0\rangle\langle 0|$. This property makes us the favour of rendering the $\mathrm{U}(\infty-1)$ part of the unbroken symmetry irrelevant as far as the four-dimensional physics is concerned. Furthermore, the modes χ_{n0} with $n \geq 1$ show up with the same terms in the limit of infinite non-commutativity. This is due to the residual $\mathrm{U}(\infty - 1)\otimes \mathrm{U}(1)$ symmetry left unbroken by the soliton (it is easy to see that the generators R_{00} and R_{mn}, S_{mn} with $m, n \geq 1$ leave the term $m^2(\Phi)\star\chi\star g(\phi_*-\Phi)\star\chi$ invariant upon integration over the two non-commutative directions). The 4D spectrum contains a real zero mode χ_{00} plus a tower of (arbitrarily) massive complex states χ_{n0} $(n \geq 1)$ which are degenerate in the limit of infinite non-commutative parameter. It is also remarkable that the localization of the massless mode $\chi_{00}(x^a)$ is due entirely to its kinetic term. This originates from the property (2.10): any function $f(\Phi)$ in front of the kinetic term is proportional to the gaussian soliton ϕ_0 which acts as a warp-factor. Furthermore, the 4D action inherits informations about the non-commutative soliton only in the point $\Phi = \phi_*$. No detailed knowledge of the function $f(\Phi)$ is needed and the 4D action for the massless mode could be rewritten as

$$S = \int d^4x\, d^2z\, \delta^{(2)}(z - z_*) f(\Phi)\, \frac{1}{2}\, \partial_a\chi_{00}\partial^a\chi_{00}\,, \tag{4.5}$$

where z_* is the point in the $\mathbb{R}^2_\star$ space where the soliton Φ_0 takes the value $\phi_0 = \phi_*$. This is quite surprising if we think that the soliton spreads in the $\mathbb{R}^2_\star$ over distances $\sim \sqrt{\Theta} \gg M_f^{-1}$. We believe this is another manifestation of the UV/IR connection pointed out first in [4]. One can also adopt a more stringy point of view and interpret the localization of the scalar field as modes that appear from nothing due to the presence of the noncommutative soliton [11, 12].

4.2 The four-dimensional action with a zero mode and a tower of Kaluza-Klein states

Let us now take the following simple quadratic action for the field χ

$$\begin{aligned} S &= \int d^4x\, d^2z \left(\frac{1}{2} f(\Phi) \star \partial_\mu\chi \star \partial^\mu\chi\right) \\ &= \int d^4x\, d^2z \left[\frac{1}{2} f(\Phi) \star (\partial_a\chi \star \partial^a\chi \;-\; \partial_z\chi \star \partial_{\bar{z}}\chi - \partial_{\bar{z}}\chi \star \partial_z\chi)\right]. \end{aligned} \tag{4.6}$$

Performing again the expansion (4.2) and neglecting the finite Θ-effects, the 4D action reads

$$S = \pi\Theta \int d^4x \left(\frac{1}{2} f(\phi_*)\partial_a\chi_{00}\partial^a\chi_{00} + \frac{1}{2}\sum_{n\geq 1} f(\phi_*)\partial_a\chi_{0n}\partial_a\chi_{n0}\right). \tag{4.7}$$

The finite Θ correction comes from the kinetic $\mathcal{L}_K$ term containing the derivatives along the non-commutative coordinates. The latter can be written as (in the rescaled coordinates)

$$
\begin{aligned}
\mathcal{L}_K &= -\frac{f(\phi_*)}{2\Theta}\int d^2w \frac{1}{2}\Phi_0 \star (\partial_w \chi \star \partial_{\bar{w}}\chi + \partial_{\bar{w}}\chi \star \partial_w \chi) \\
&= -\frac{\pi}{\Theta} f(\phi_*)\frac{1}{2}\sum_{m,n,r,s\geq 0} \chi_{mn}\chi_{rs}\mathrm{Tr}\ \{\Phi_0 [a, \Phi_{mn}] [\Phi_{rs}, \bar{a}] + \Phi_0 [\Phi_{rs}, \bar{a}] [a, \Phi_{mn}]\} .
\end{aligned}
\tag{4.8}
$$

Using the relations

$$
\begin{aligned}
[a, \Phi_{mn}] &= \sqrt{m}\Phi_{m-1n} - \sqrt{n+1}\Phi_{mn+1}, \\
[\Phi_{rs}, \bar{a}] &= \sqrt{s}\Phi_{rs-1} - \sqrt{r+1}\Phi_{r+1s} ,
\end{aligned}
\tag{4.9}
$$

we find the kinetic term $\mathcal{L}_K$

$$
\begin{aligned}
\mathcal{L}_K = -\frac{\pi f(\phi_*)}{\Theta}\Big[&(\chi_{00}-\chi_{11})^2 + 4\chi_{10}\chi_{01} - \sqrt{2}\chi_{01}\chi_{21} - \sqrt{2}\chi_{10}\chi_{12} + \chi_{12}\chi_{21} + \\
&+\sum_{n\geq 2}(2n+1)\chi_{n0}\chi_{0n} - \sum_{n\geq 3}\sqrt{n}\chi_{0,n-1}\chi_{n1} - \sum_{n\geq 3}\sqrt{n}\chi_{1n}\chi_{n-1,0} + \sum_{n\geq 3}\chi_{1,n}\chi_{n1}\Big] .
\end{aligned}
\tag{4.10}
$$

It is clear from (4.10) that the degrees of freedom associated with χ_{1n} with $n \geq 1$ do not propagate, indeed the corresponding equations of notions lead to the following algebraic constraints

$$
\chi_{1n} = \sqrt{n}\chi_{0n-1} , \qquad n \geq 1 . \tag{4.11}
$$

The effective action for the propagating degrees of freedom takes the form

$$
\begin{aligned}
S = \pi\Theta \int d^4x \Big\{ &\frac{1}{2} f(\phi_*)\partial_a\chi_{00}\partial^a\chi_{00} + \sum_{n\geq 1}\Big[\frac{1}{2}f(\phi_*)\partial_a\chi_{0n}\partial^a\chi_{n0} - \\
&- \frac{1}{2}f(\phi_*)(n+1)\Theta^{-1}\chi_{0n}\chi_{n0}\Big]\Big\} .
\end{aligned}
\tag{4.12}
$$

The kinetic term $\mathcal{L}_K$ provides all the modes $\chi_{0n}(x^a)$ with $n \geq 1$ with an extra mass squared $\sim (n+1)f(\phi_*)/\Theta$. Therefore, the four-dimensional spectrum contains[3] a massless real state $\chi_{00}(x^a)$ and an infinite tower of massive complex modes $\chi_{0n}(x^a)$ with mass squared and spacing proportional to $1/\Theta \gg \mathrm{TeV}^2$! The four-dimensional action therefore looks like a 5D Kaluza-Klein theory compactified on a circle of radius $\sqrt{\Theta}$. This phenomenon arises from the fact that the degrees of freedom propagating in the two extra dimensions, due to the commutation relations, actually correspond

[3] A similar computation shows that the same result applies to the massless modes $\varphi_{0n}(x^a)$ of the fluctuations of the soliton in the case in which the kinetic term is multiplied by a function $f(\Phi)$ (and the $\mathrm{U}(\infty)$ symmetry is not gauged). This is equivalent to say that the soliton Φ_0 is stable against small fluctuations around it. Of course, the modes φ_{01} and its complex conjugate φ_{10} remain massless since they represent the translational zero modes arising from the spontaneous breaking of translation invariance along the directions x^4 and x^5.

to a point-particle in one dimension. This is a remarkable result. Not only the presence of the soliton Φ_0 projects out all the modes χ_{mn} with $m, n \geq 1$ from the 4D effective theory, but also leaves behind a single massless mode plus a tower of very heavy KK-states. This happens in spite of the fact that the two extra dimensions we started from are infinite. Again, the localization of the massless mode $\chi_{00}(x^a)$ is due entirely to its kinetic term.

4.3 The four-dimensional action with the zero mode only

Consider now the following action for the scalar field χ

$$S = \int d^4x d^2z \frac{1}{2} f(\Phi) \star \partial_\mu \chi \star g(\Phi) \star \partial^\mu \chi \,. \tag{4.13}$$

In the infinite limit of the non-commutative parameter, the four-dimensional action reads

$$S = \pi\Theta \int d^4x \frac{1}{2} f(\phi_*) g(\phi_*) \partial_a \chi_{00} \partial^a \chi_{00} \,. \tag{4.14}$$

It is easy to see that the kinetic term $\mathcal{L}_K$ depending on the non-commutative coordinates gives rise to a 4D action of the type $\int d^4x \, |\chi_{01}|^2$ and — since the fields χ_{01} and χ_{10} do not propagate — it vanishes identically. Therefore, all modes but the zero mode χ_{00} have been projected out! The four-dimensional action has lost completely the notion of being immersed in the extra-dimensional world.

5. Localization of chiral fermions

In this section we sketch how to extend the previous analysis to fermions. The Dirac matrices which generate the Clifford algebra are taken to be

$$\Gamma^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix}, \qquad \Gamma^4 = i \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \tag{5.1}$$

We define the chirality operator

$$\overline{\Gamma} = \Gamma^0 \Gamma^1 \cdots \Gamma^5 = \begin{pmatrix} -\gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix}, \tag{5.2}$$

which has the properties

$$\left(\overline{\Gamma}\right)^2 = 1 \,, \qquad \left\{\overline{\Gamma}, \Gamma^\mu\right\} = 0. \tag{5.3}$$

A Dirac spinor in six dimensions has 8 complex components and can decomposed as the direct sum of a pair of 6D Weyl spinors

$$\Psi = \begin{pmatrix} \zeta \\ \psi \end{pmatrix}, \tag{5.4}$$

where each Weyl spinor corresponds to a Dirac spinor in 4D. One can also define two 6D Weyl spinors

$$\Psi_- = \frac{1-\overline{\Gamma}}{2}\Psi = \begin{pmatrix} \zeta_R \\ \psi_L \end{pmatrix}, \qquad \Psi_+ = \frac{1+\overline{\Gamma}}{2}\Psi = \begin{pmatrix} \zeta_L \\ \psi_R \end{pmatrix}, \tag{5.5}$$

each of them representing a Dirac spinor in 4D. Each spinor can be decomposed as usual as

$$\zeta(x^\mu) = \sum_{m,n\geq 0} \zeta(x^a)\Phi_{mn}(x^i)\,, \qquad \psi(x^\mu) = \sum_{m,n\geq 0} \psi(x^a)\Phi_{mn}(x^i)\,. \tag{5.6}$$

Since the generator of the isometry group U(1) which is left unbroken by the soliton Φ_0 is given in polar coordinates (r,θ) by $(\partial_\theta + (1/2)i\tau_3)$[4] and $i\partial_\theta$ in the operator language is proportional to $(\overline{a}a + a\overline{a})/2$, it easy to see that the fermions ζ_{n0} and ζ_{0n} have charge $n+(1/2)$ and $-n+(1/2)$ respectively under the isometry group U(1), while the fermions ψ_{n0} and ψ_{0n} have charge $n-(1/2)$ and $-n-(1/2)$, respectively.

For the sake of simplicity, we consider the following action in the limit of infinite non-commutativity

$$\begin{aligned} S &= \int d^6x \left[\frac{i}{2} f(\Phi)\star\left(\overline{\Psi}\star g(\Phi)\star\Gamma^\mu\partial_\mu\Psi - \partial_\mu\overline{\Psi}\star g(\Phi)\star\Gamma^\mu\Psi\right)\right] \\ &= \int d^6x \left(\frac{i}{2} f(\Phi)\star\overline{\Psi}\star g(\Phi)\star \overleftrightarrow{\partial_\mu}\, \Gamma^\mu\Psi\right); \end{aligned} \tag{5.7}$$

with $f(0) = g(0) = 0$. The 4D action reads

$$S = \Theta\pi \int d^4x f(\phi_*)g(\phi_*)\left(\overline{\psi}_{00}\gamma^a \overleftrightarrow{\partial_a}\, \psi_{00} + \overline{\zeta}_{00}\gamma^a \overleftrightarrow{\partial_a}\, \zeta_{00}\right). \tag{5.8}$$

We are left with a four-dimensional action containing two 4D Dirac massless spinors whose localization is due once more to its kinetic term and to the role of warp factor played by the soliton. The relevant point though is that – had we started from a Weyl spinor in 6D – we would have ended up with a chiral theory! Indeed, suppose we start with only the Weyl spinor Ψ_- (Ψ_+). The 4D theory then contains the two Weyl 4D spinors ψ_L^{00} and ζ_R^{00} (ψ_R^{00} and ζ_L^{00}). Since ψ_L^{00} (ψ_R^{00}) is a spinor with charge $-\frac{1}{2}$ and ζ_R^{00} (ζ_L^{00}) its with charge $\frac{1}{2}$, the theory admits chiral fermions with respect to the isometry group U(1) which is left unbroken by the presence of the noncommutative soliton.

Of course and in complete similarity with the scalar case analyzed in the previous section, different choices of the starting action other than (5.7) lead to different fermionic spectra.

[4]Notice that the commutation relations between z and $\overline{z}$ are invariant under translations and rotations and spinors have definite properties under these transformations.

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6. Localization of gauge fields

As we have seen in section 3, the soft fluctuations around the soliton $\Phi = \phi_* \Phi_0$ are made of a tower of massless states φ_{0n} with $n \geq 1$ in the limit of infinite non-commutative parameter. They reflect the symmetry breakdown of $\mathrm{U}(\infty)$ symmetry down to $\mathrm{U}(\infty - 1) \otimes \mathrm{U}(1)$ caused by the presence of the soliton Φ_0 in $\mathbb{R}^2_\star$. These Goldstone are not necessarily problematic from the four-dimensional point of view as they may acquire a mass through the finite Θ-effects induced by the kinetic term along the two non-commutative directions.

There is an alternative route one may take, though. If the global $\mathrm{U}(\infty)$ symmetry is promoted to a gauge symmetry, the latter becomes a $\mathrm{U}(1)$ gauge symmetry of the non-commutative theory and removes from the spectrum the unwanted Goldstone bosons; they are simply eaten by the Higgs mechanism.

This procedure has another advantage. It leaves the four-dimensional effective action only with a massless $\mathrm{U}(1)$ gauge field[5] (plus the components of the gauge fields transverse to the soliton which behave as adjoint Higgs fields – the graviphotons – on the soliton and play the role of the Goldstone modes from the breaking of the translation invariance along the (x^4, x^5)-directions).

Therefore, the presence of the radially symmetric soliton Φ_0 in $\mathbb{R}^2_\star$ and the corresponding breaking of the gauge symmetry $\mathrm{U}(\infty)$ provides a quantum field-theoretic mechanism to localize a $\mathrm{U}(1)$ gauge field onto a four-dimensional brane starting from a six-dimensional world with infinite extra-dimensions. This does not come as a surprise as our procedure is the quantum field-theoretic analog of the one adopted by Harvey et *al.* in ref. [13] where — in a string-theoretic context — it was shown that the non-commutative geometry induced by a large auxiliary magnetic field allows the identification of D-branes in open string theory with the non-commutative tachyonic solitons.

6.1 Promoting the global $\mathrm{U}(\infty)$ symmetry to a gauge symmetry

The gauge transformation law of a non-commutative gauge field A_μ is given by

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon - i[A_\mu, \epsilon]\,, \tag{6.1}$$

where

$$[A_\mu, \epsilon] = A_\mu \star \epsilon - \epsilon \star A_\mu\,. \tag{6.2}$$

The corresponding field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\,. \tag{6.3}$$

When non-commutativity is present, even the transformation law for an abelian $\mathrm{U}(1)$ gauge field is nontrivial. This happens because the commutator of two infinitesimal

[5]For alternative higher-dimensional mechanisms to localize gauge fields, see [15, 16].

gauge transformations with generators ϵ_1 and ϵ_2 a gauge transformation generated by $i(\epsilon \star \epsilon_2 - \epsilon_2 \star \epsilon)$. Such commutators are nontrivial in the case of a gauge group U(N) with rank 1, i.e. $N = 1$, even though in the limit $\Theta = 0$ the rank 1 case is the abelian U(1) gauge group.

We now assume that the scalar field Φ transforms in the adjoint of a non-commutative U(1):

$$\delta_\epsilon \Phi = -i[\Phi, \epsilon]\,, \tag{6.4}$$

so that the covariant derivative reads

$$D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi]\,. \tag{6.5}$$

The 6D action is then

$$S = \int d^6x \left[\frac{1}{2} f(\Phi) D^\mu \Phi D_\mu \Phi - V(\Phi) - \frac{1}{4} h(\Phi) F^{\mu\nu} F_{\mu\nu}\right]; \tag{6.6}$$

with $f(0) = h(0) = 0$. Again, we take first the limit of infinite non-commutative parameter. The derivatives along the non-commutative directions x^i are all suppressed, yielding

$$\begin{aligned} S = \int d^6x \Big[&\frac{1}{2} f(\Phi) D^a \Phi D_a \Phi - \frac{1}{2} h(\Phi) D^a A_i D_a A^i - V(\Phi) - \\ &- \frac{1}{2} f(\Phi)[A_i, \Phi][A^i, \Phi] + \frac{1}{4} h(\Phi)[A_i, A_j][A^i, A^j] - \frac{1}{4} h(\Phi) F^{ab} F_{ab}\Big]\,. \end{aligned} \tag{6.7}$$

In the limit of infinite non-commutativity, the A_i ($i = 4, 5$) become adjoint scalar fields and the action becomes invariant under the gauge transformations

$$\begin{aligned} \delta_\lambda \Phi &= -i[\Phi, \lambda]\,, \\ \delta_\lambda A_i &= -i[A_i, \lambda]\,, \\ \delta_\lambda A_a &= \partial_a \lambda - i[A_a, \lambda]\,. \end{aligned} \tag{6.8}$$

Using now the correspondence (2.3), the action (6.7) appears from the four-dimensional point of view as the action of a U(∞) gauge theory coupled to the adjoint scalars Φ and A_i (for instance, the infinitesimal transformation (6.8) of the field Φ is the infinitesimal form of the transformation $\hat{\Phi} \to U \hat{\Phi} U^\dagger$). The gauge symmetry remains exact even for finite non-commutativity [13] provided that the derivatives $\partial_i \epsilon$ reappear in $\delta_\epsilon A_i$.

6.2 The fluctuations of the gauge fields around the soliton

Since the action (6.7) is invariant under a gauge U(∞) symmetry, we make use of the gauge freedom to work in the unitary gauge along which the fluctuations around the soliton (3.3) are such that

$$\varphi_{0n} = \varphi_{n0} = 0 \qquad \text{for} \qquad n \geq 1\,. \tag{6.9}$$

The fluctuations of the gauge fields can be decomposed as usual as

$$\begin{aligned} A_a(x^\mu) &= \sum_{m,n\geq 0} A_a^{mn}(x^a)\Phi_{mn}(x^i)\,, \\ A_i(x^\mu) &= \sum_{m,n\geq 0} A_i^{mn}(x^a)\Phi_{mn}(x^i)\,, \end{aligned} \tag{6.10}$$

where A_a^{mn} and A_i^{mn} represent hermitian matrices.

Plugging these fluctuations into the action (6.7) and computing the trace, we find that all modes with $m, n \geq 1$ are again projected out by the soliton background at the quadratic order. What is left in the spectrum are the fields φ_{00}, A_a^{00}, A_i^{00}, together with the $0m$- and $m0$-components of fields A_i and A_a.

The action for the 00-components of the fields is given by

$$\begin{aligned} S = \pi\Theta \int d^4x \bigg[&\frac{1}{2} f(\phi_*)\partial^a\varphi_{00}\partial_a\varphi_{00} - \frac{1}{2}V''(\phi_*)\varphi_{00}^2 + \frac{1}{2}h(\phi_*)\partial^a A_i^{00}\partial_a A_i^{00} - \\ &- \frac{1}{4}h(\phi_*)F^{ab}F_{ab} \bigg]\,, \end{aligned} \tag{6.11}$$

where $F_{ab} = \partial_a A_b^{00} - \partial_b A_a^{00}$ is the standard field strength. This is the 4D action describing a massless U(1) gauge field plus the two transverse scalars A_i^{00} which play the role of the Goldstone modes arising from the spontaneous breakdown of the translation invariance along the (x^4, x^5)-directions. It is easy to understand why the U(1) gauge field A_a^{00} remains massless; there is simply no Goldstone mode φ_{00} to be eaten. The latter is indeed massive, see eq. (3.6). Another way of thinking of it is that A_a^{00} is the massless gauge field corresponding to the residual U(1) symmetry (or rotational isometry) which is left unbroken in the spontaneous breakdown process $U(\infty) \to U(\infty - 1) \otimes U(1)$. Furthermore, the fact that the symmetry is a gauge symmetry guarantees that the field A_a^{00} remains massless even when finite Θ-effects are taken into account .[6].

To see explicitly how the action (6.11) comes about, take for example the term

$$\int d^6x\, f(\Phi)[A_\mu, \Phi][A^\mu, \Phi] \tag{6.12}$$

present in the action (6.7). Using the relation $f(\phi_*\Phi_0) = f(\phi_*)\Phi_0$ and carrying out the trace, it gives rise to

$$\int d^4x\, f(\phi_*)\phi_*^2 \left(A_\mu^{00} A_{00}^\mu - \sum_{n\geq 0} A_\mu^{0n} A_{n0}^\mu \right) = -\sum_{n\geq 1} \int d^4x\, f(\phi_*)\phi_*^2\, A_\mu^{0n} A_{n0}^\mu\,. \tag{6.13}$$

[6]Similarly, had we started from a level k soliton solution (2.16), the resulting unbroken gauge symmetry would have been the non-abelian U(k) symmetry.

This shows explicitly how the modes A_a^{00}, A_i^{00} remain massless. On the other hand, the $0m$ and $m0$ components of fields A_μ acquire a mass

$$m_A^2 = \frac{f(\phi_*)}{h(\phi_*)}\phi_*^2 \tag{6.14}$$

being the corresponding quadratic action

$$S = \pi\Theta \sum_{n\geq 1} \int d^4x \left[\frac{1}{2}h(\phi_*)\partial^a A_i^{0m}\partial_a A_{m0}^i - \frac{1}{2}f(\phi_*)\phi_*^2 A_i^{0m} A_{m0}^i\right]. \tag{6.15}$$

This is the announced Higgs mechanism. Thus, we find the remarkable result that — starting from a six dimensional theory with infinite extra dimensions — the non-commutative soliton projects out all the undesired gauge field states leaving behind only one massless photon (plus the graviphotons) whose localization on the soliton is only due to its kinetic term. As a final comment we note that if we insist on having a 4D weakly coupled gauge theory, due the θ enhancement coming from integrating out the extra dimensions, the 6D theory from which we started is strongly coupled, this resembles what happens in [15].

7. Conclusions and perspectives

In this paper we have examined the construction of what we dubbed the non-commutative world brane. The underlying theory is a six-dimensional non-commutative field theory where non-commutativity shows up along the two extra infinite spatial dimensions. The four-dimensional world-volume is a scalar soliton living in the non-commutative space. It appears that such a construction provides a mechanism to localize scalar, fermionic and gauge fields. Let us now consider the advantages and the possible obstacles to such a mechanism.

First, the advantages: the study of localization of fields inside the non-commutative scalar soliton is made easy by the powerful correspondence between non-commutative coordinates and operators on a single particle Hilbert space. Equations of motion in field theory translate into algebraic operator equations and integrals over the two extra dimensions reduce to simple traces over operators expressed in terms of the eigenstates of a simple one-dimensional oscillator system. Under this correspondence, the non-commutative soliton behaves like a projector operator and deriving the spectrum of the low-energy four-dimensional action is rather simplified. Field localization takes place through the coupling to the soliton in the kinetic term and — thanks to the crucial property (2.10) — the soliton acts like a warp factor which is quite insensitive to the form of the coupling itself. Yet changing the latter gives rise to diverse low-energy spectra. On a more phenomenological side, our construction admits chiral fermionic zero modes and it ensures the localization of massless gauge fields very much the same as it happens in string theory where gauge fields (interpreted as the end-points of open strings) get localized on D-branes identified with non-commutative tachyonic solitons when a large auxiliary magnetic field is turned on.

Now the possible obstacles. At present, we do not know how gravity behaves in the presence of the non-commutative soliton and if four-dimensional observers may still empirically deduce a four-dimensional Newton's law. In fact, the study of gravity in non-commutative spaces is just as its infancy. In non-commutative spaces the metric becomes complex and one can obtain complexified gravity in D dimensions by gauging the symmetry $\mathrm{U}(1, D-1)$ instead of the usual $\mathrm{SO}(1, D-1)$ [17]. The metric contains the usual symmetric part plus an antisymmetric tensor whose origin traces back to the fact that the non-commutativity antisymmetric parameter $\Theta_{\mu\nu}$ becomes a dynamical field. This might not be necessarily a threat to our construction since non-commutativity shows up only in the two extra dimensions. Furthermore, one can write a both unique and gauge invariant gravity action [17]. Clearly, all these issues deserve to be addressed and we are now working along these directions. What makes us hopeful is that the study of the gravitational background and its excitations in the presence of the non-commutative soliton is to some extent simplified by the nice properties of the non-commutative soliton and by its capability of playing the role of a warp factor.

It would be also worth considering what are the cosmological implications of our set-up. Because of the UV/IR mixing, fluctuations of a scalar field at small scales ℓ reappear at large distances Θ/ℓ [4] and it has been recently shown that solitons can travel faster than the speed of light for arbitrarily long distances [18]. These ingredients may be relevant to construct an alternative to the theory of inflation [19] and the generation of the cosmological density perturbations [20]. Another interesting project might be to investigate if there are new types of four-dimensional topological defects whose core might act as windows to the extra dimensions [21] and thus to the non-commutative world.

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